Equations (23, 36, and 37) describe the desired spiral trajectory. As is seen from Eq. (36), the spiral trajectory is of oscillatory nature, which damps out as the trajectory progresses. If the second term in Eq. (36) is dropped out because of $A \gg 1$, Eq. (36) yields

$$u = 1/p^2 \tag{38}$$

In combining with Eq. (23), Eq. (38) becomes

$$u = \{\omega(\bar{c}/g_0) \ln[1 - (\tau/A)] + 1\}^2 \tag{39}$$

which is the result obtained by Melbourne.[4]

Conclusions

Briefly, application of the stroboscopic method in nonlinear mechanics has been shown to provide concise solutions to the spiral trajectory under low constant tangential thrust previously discussed in the literature. The oscillatory nature of the spiral is brought out. The oscillations cause a maximum deviation from the mean path of the order of $2/A$ during the early stage of spiraling and gradually damp out as spiral proceeds. Figure 2 shows such an oscillatory spiral trajectory for a space vehicle ($\bar{c}$ = 161,000 fps), which is initially in a 300-naut-mile orbit around the earth and is subject to a constant tangential thrusting force of $0.005 M_0 g_0$.

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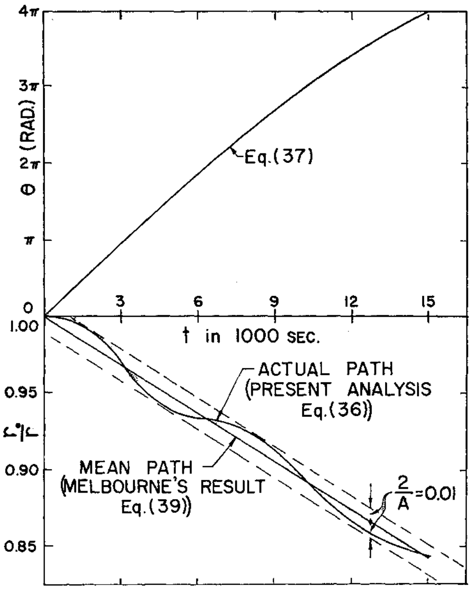

Fig. 2 Oscillatory spiral trajectory

[2] Minorsky, N., *Nonlinear Oscillations* (D. Van Nostrand Co. Inc., Princeton, N. J., 1962), Chap. 14, pp. 329–335.

[3] Minorsky, N., *Nonlinear Oscillations* (D. Van Nostrand Co. Inc., Princeton, N. J., 1962), Chap. 16, pp. 390–415.

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Stability Boundaries of Liquid-Propelled Space Vehicles with Sloshing

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For liquid-propelled space vehicles with large diameter propellant containers, the effects of propellant sloshing upon the vehicle stability are becoming more critical, especially since at launch a very large amount of the total weight is in form of liquid propellant. Describing the propellant motion with a mechanical model, the influence of propellant sloshing in one, two, and three tanks is determined with the Hurwitz stability boundaries. They are represented as the necessary required damping in the containers vs the container location. The influence of the various parameters, such as control frequency and control damping of the vehicle, as well as sloshing mass, sloshing frequency, and tank locations, is investigated. For a space vehicle in which the propellant of only one tank is free to oscillate, the danger zone where baffles may have to be applied to maintain stability is located between the center of gravity and the center of instantaneous rotation. The influence of tank geometry such as concentric containers or a quarter-tank arrangement is investigated also. For a tandem arrangement of two and three containers in which the propellant is free to oscillate, the danger zone shifts with increasing distance between the slosh masses toward the rear of the space vehicle, indicating in a practical case that most of the booster tanks of a space vehicle have to be provided with appropriate baffles to maintain vehicle stability.

I. Introduction

THE motion of the liquid propellant in the tanks of a space vehicle represents, due to its low natural frequencies that are usually very close to the control frequency, a potential hazard for stability and control. The stability of a space vehicle can be influenced tremendously by this propellant sloshing. Very useful results can be obtained from simplified stability investigations of a rigid space vehicle that is controlled by a simple control system. By variation of parameters, the possibility of the destabilizing effect of the propellant is investigated. The propellant motion in the containers will be described by the mechanical model as derived in Ref. 1. The stability boundaries are given by the necessary damping values of the propellant along the vehicle.

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Tank arrangement and tank form may enhance the stability behavior of the vehicle. In order to obtain a general knowledge of the influence of tank location and tank forms, the interaction between the translation, pitching, and propellant sloshing is treated. The influence of tank geometry (which essentially determines the magnitude of the modal masses and the eigenfrequencies of the liquid), the location of the tank, and the various gain values of the control system are investigated. To simplify the computation, aerodynamic effects and the inertia of the swivel engines are neglected. As shown in Refs. 1 and 2, the first modal mass of the liquid is sufficient in circular symmetric containers to describe the motion of the propellant. Higher modal masses have practically no influence on stability. This is justified because in a circular cylindrical tank the ratio of the sloshing mass to the propellant mass for the second sloshing mode is already smaller than 3% of the first sloshing mode. In a cylindrical tank with annular cross section, the second sloshing mode mass is, in the most unfavorable case for a diameter ratio of inner-to-outer diameter $k = 0.5$, less than about 12%.[3] The two lowest modal masses have to be considered only in the four-quarter tank arrangement. This seems necessary because the second modal mass no longer can be neglected with respect to the first one; it is about 43% of the modal mass of the first mode. Higher modes in the quarter tank arrangement have sloshing masses that are below 7% of the first mode and can be neglected.

The control moments will be produced by swivel engines. In the following, a space vehicle with a simple control system is treated. The coordinate system has its origin at the center of gravity of the undisturbed vehicle. The accelerated coordinate system is substituted by an inertial system such that the space vehicle is subjected to an equivalent field of acceleration (Fig. 1). The centrifugal and Coriolis forces, which result from rotation, will be neglected. Furthermore, the acceleration in the direction of the trajectory, the mass, the moment of inertia, etc. will be considered constant. The parameters that effect the stability of the spacecraft are the ratios of oscillating propellant mass to the total vehicle mass of each tank, the natural slosh frequencies, the control frequency, the control damping, the gain factor of the attitude control system, the diameter ratio of the inner-to-outer tank in case of an annular cylindrical tank, and distance between the slosh masses of various tanks.

II. Equations of Motion

The equation of motion perpendicular to the trajectory is, with m as the total mass of the space vehicle, m_ν as the modal mass of the propellant, φ as the deviation of the attitude angle, F as the thrust, and β as the swivel angle of the engines,

$$m\ddot{y} + \sum_\nu m_\nu \ddot{y}_\nu - F\varphi - \lambda F\beta = 0 \tag{1}$$

The factor λ in the control term is because only the λth part of the thrust is available for control purposes. The displacement of the model mass, m_ν, of the propellant is denoted by y_ν. The transversal displacement of the space vehicle from its trajectory is given by y.

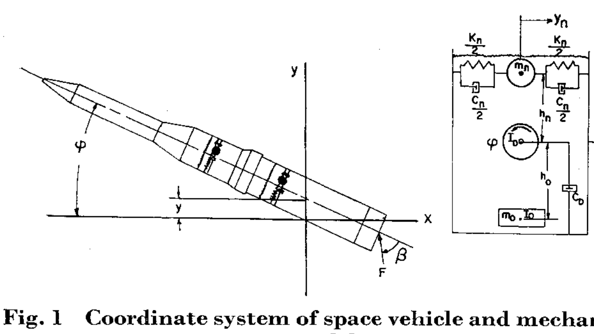

Fig. 1 Coordinate system of space vehicle and mechanical model

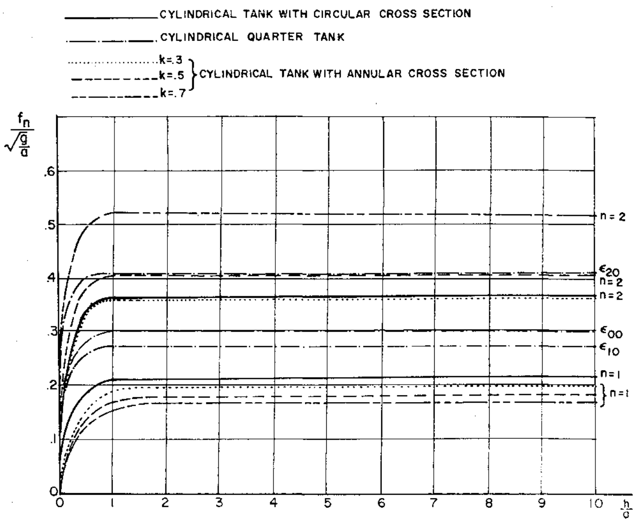

Fig. 2 Natural frequencies

The mass of the structure and the propellant (in those containers in which the sloshing is not considered), being considered as rigid, is denoted by m_a' for unit length. I_0' is the moment of inertia of the structure about the center of gravity of the cross section, $m_{0\nu}$ is the nonsloshing mass of the liquid in the container ν, and $x_{0\nu}$ is its distance from the center of gravity. Furthermore, $I_{0\nu}$ is the moment of inertia of the nonsloshing mass about its center of gravity, and x_ν is the location of the modal masses. Considering I as the effective moment of inertia of the vehicle about its center of gravity, the equation of motion with respect to pitching (with respect to center of gravity of the vehicle) is obtained.

$$I\ddot{\varphi} + \lambda F x_E \beta + \sum_\nu (m_\nu x_\nu \ddot{y}_\nu - g m_\nu y_\nu) = 0 \tag{2}$$

Here, it is

$$I = \int (x^2 m_a' + I_0')\, dx + \sum_\nu (m_{0\nu} x_{0\nu}^2 + I_{0\nu}) + \sum_\nu m_\nu x_\nu^2 \equiv mk^2$$

The term x_E is the distance of the swivel point of the engines from the origin, and k is the radius of gyration of the vehicle. In the further treatment, it is useful to refer all lengths to the radius of gyration, $k = (I/m)^{1/2}$.

The equation of modal sloshing mass is

$$\ddot{y}_\nu + 2\omega_\nu \gamma_\nu \dot{y}_\nu + \omega_\nu^2 y_\nu - x_\nu \ddot{\varphi} - g\varphi + \ddot{y} = 0 \tag{3}$$

where ν represents either or both the number of containers and the various modes of the liquid in the containers. The term ω_ν represents the natural circular frequency of the liquid, and γ_ν represents the damping factor. The longitudinal acceleration is $g = F/m$. For convenience, a few of the main results for the propellant will be given here.

A. Cylindrical Tank with Circular Cross Section

For a cylindrical tank of circular cross section, the natural frequencies are (Fig. 2)

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\left[\frac{g\epsilon_n}{a}\tanh\left(\epsilon_n \frac{h}{a}\right)\right]^{1/2} \qquad n = 1, 2, 3 \ldots$$

where a is the radius of the tank, h is the propellant height, and g is the longitudinal acceleration of the vehicle. The ϵ_n's are the roots of the first derivative of the Bessel function of first order and first kind $J_1'(\epsilon_n) = 0$. The mass m^ν of the sloshing propellant is obtained from

$$m_\nu = m_p \frac{2\tanh[\epsilon_\nu(h/a)]}{[\epsilon_\nu(h/a)](\epsilon_\nu^2 - 1)} \qquad \nu = 1, 2, 3 \ldots$$

where m_p is the mass of the propellant in the tank (Fig. 3).

B. Circular Cylindrical Tank with Annular Cross Section

For a cylindrical tank of annular cross section and diameter ratio k of the inner-to-outer tank, the natural frequencies are[2] (Fig. 2)

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\left[\frac{g\xi_{n-1}}{a}\tanh\left(\xi_{n-1}\frac{h}{a}\right)\right]^{1/2} \quad n = 0, 1, 2, 3$$

where a is the radius of the outer tank and the ξ_n's are the roots of the determinant

$$\Delta_1(\xi) = \begin{vmatrix} J_1'(\xi) & Y_1'(\xi) \\ J_1'(k\xi) & Y_1'(k\xi) \end{vmatrix} = 0$$

$k = b/a$ being the ratio of the inner-to-outer tank radius. The sloshing mass is

$$m_\nu = \frac{\bar{A}_\nu m_p[(2/\pi\xi_\nu) - k\ C_1(k\xi_\nu)]\tanh[\xi_\nu(h/a)]}{(1-k^2)[\xi_\nu(h/a)]}$$

where

$$\bar{A}_\nu = \frac{2[(2/\pi\xi_\nu) - k\ C_1(k\xi_\nu)]}{(4/\pi^2\xi_\nu^2)(\xi_\nu^2 - 1) + C_1^2(k\xi_\nu)(1 - k^2\xi_\nu^2)}$$

and

$$C_1(k\xi_\nu) = \begin{vmatrix} J_1(k\xi_\nu) & Y_1(k\xi_\nu) \\ J_1'(\xi_\nu) & Y_1'(\xi_\nu) \end{vmatrix}$$

The term m_p is the liquid mass in the annular tank.

C. Four Circular Cylindrical Quarter Tanks

For four cylindrical quarter tanks with radius a and filled with liquid to a height h, the natural frequencies of the propellant are[2]

$$f_{mn} = \frac{1}{2\pi}\left[\frac{g\epsilon_{mn}}{a}\tanh\left(\epsilon_{mn}\frac{h}{a}\right)\right]^{1/2} \quad \begin{matrix} n = 1, 2, 3\ldots \\ m = 0, 1, 2\ldots \end{matrix}$$

where the ϵ_{mn}'s are the roots of $J_{2m}'(\epsilon) = 0$ (Fig. 2). The mass of the sloshing propellant is obtained from

$$m_{mn} = m_p \frac{64\ \epsilon_{mn}\tanh[\epsilon_{mn}(h/a)]}{\pi^2(\epsilon_{mn}^2 - 4m^2)[\epsilon_{mn}(h/a)]J_{2m}^2(\epsilon_{mn})} \cdot \left[\frac{J_{2m}(\epsilon_{mn})}{(4m^2-1)} + \frac{2}{\epsilon_{mn}}\sum_{\mu=0}^{\infty} J_{2m+2\mu+1}(\epsilon_{mn})\right] \cdot \sum_{\mu=0}^{\infty}\frac{J_{2m+2\mu+1}(\epsilon_{mn})}{(2m+2\mu-1)\cdot(2m+2\mu+3)}$$

(for $m = 0$ the coefficient 64 is substituted by 32) where m_p is the mass of the propellant in the tank. It can be seen that one more mode must be considered (Fig. 3) because $m_{00} \approx 0.43m_{10}$.

Actually, the control equation cannot be written as a linear equation. However, translatory and rotational vibrations usually occur at small frequencies where the control elements can be considered as essentially linear. Nonlinearities usually occur at higher frequencies in the form of saturation of amplifiers, limited output of velocities, etc. Without wind disturbances, the control equation can be written in the form, $f_1(\beta) = f_2(\varphi)$, where f_1 and f_2 are functions depending on the character of the system. In linear form one can write these as

$$\textstyle\sum_\nu p_\nu\beta^{(\nu)} = \sum_\lambda a_\lambda\varphi^{(\lambda)}$$

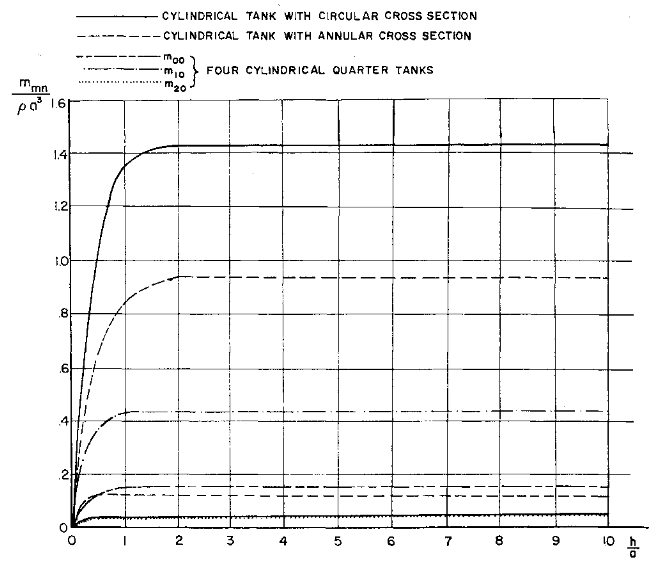

Fig. 3 Sloshing mass

Here, p_ν's are the so-called phase lag coefficients, and φ is the indicated deviation from the trajectory as indicated by the gyro. One can apply

$$\beta = a_0\varphi + a_1\dot{\varphi} \tag{4}$$

In this form of the control equation, the time derivatives of the control angle β which cause, with increasing frequency, increasing phase lags have been neglected. This is justified because of the small magnitude of control and propellant frequencies.

III. Stability Boundaries of a Space Vehicle with a Simple Control System

To obtain the main results of the influence of the propellant sloshing on the stability, the equations of motion and the propellant will be treated as being free to oscillate in three tanks. This seems to be sufficient since usually, even in large space vehicles, only three of the tanks will exhibit large sloshing masses. The sloshing propellant masses of tanks with light propellant and tanks of smaller diameter can be neglected. With the usual assumption for solution of the form $e^{s\omega_c t}$, where s is the complex frequency, $s = \sigma + i\omega$, the differential equations are transformed into homogeneous algebraic equations:

$$\begin{vmatrix} s^2\omega_c^2 & -g(1+\lambda a_0+\lambda a_1 s\omega_c) & \mu_1 s^2\omega_c^2 & \mu_2 s^2\omega_c^2 & \mu_3 s^2\omega_c^2 \\ 0 & s^2\omega_c^2 + \dfrac{\lambda g x_E}{k^2}(a_0 + s\omega_c a_1) & \dfrac{\mu_1}{k^2}(x_1\omega_c^2 s^2 + g) & -\dfrac{\mu_2}{k^2}(x_2\omega_c^2 s^2 + g) & -\dfrac{\mu_3}{k^2}(x_3\omega_c^2 s^2 + g) \\ s^2\omega_c^2 & -(x_1\omega_c^2 s^2 + g) & s^2\omega_c^2 + 2\gamma_1\omega_1\omega_c s + \omega_1^2 & 0 & 0 \\ s^2\omega_c^2 & -(x_2\omega_c^2 s^2 + g) & 0 & s^2\omega_c^2 + 2\gamma_2\omega_2\omega_c s + \omega_2^2 & 0 \\ s^2\omega_c^2 & -(x_3\omega_c^2 s^2 + g) & 0 & 0 & s^2\omega_c^2 + 2\gamma_3\omega_3\omega_c s + \omega_3^2 \end{vmatrix} = 0 \tag{5}$$

Here, $\mu_\nu = m_\nu/m$ is the ratio of the sloshing mass in the νth container to the total mass of the vehicle. For nontrivial solutions, the coefficient determinant (5) must vanish with which one obtains the characteristic polynomial in s,

$$\sum_{i=0}^{8} B_i s^i = 0$$

The coefficients B_i depend on these parameters:

1) $\mu_s = m_s/m$ ratios of modal mass of liquid over total mass of space vehicle,
2) ζ_c = control damping factor,
3) $\nu_s = \omega_s/\omega_c$ frequency ratios of undamped propellant frequency to undamped control frequency,
4) γ_s = damping factor of propellant,

5) $\xi_s = x_s/k$ ratio of the coordinate of the location of the modal mass of the propellant to radius of gyration of the space vehicle,

6) a_0 = gain value of the attitude control system, and

7) ξ_1, ξ_2 = distances between rear tank and upper tanks, respectively.

The stability boundaries are characterized by the roots s, of which the real part will be zero, whereas the others are stable roots. This is, with the Hurwitz theorem for a stability polynominal of the nth degree, $B_n = 0$ and $H_{n-1} = 0$, where H_{n-1} represents the Hurwitz determinant of the form

$$H_{n-1} = \begin{vmatrix} B_1 & B_3 & B_5 & \dots \\ B_0 & B_2 & B_4 & \dots \\ 0 & B_1 & B_3 & \dots \\ 0 & B_0 & B_2 & \dots \end{vmatrix} \quad (n-1) \text{ lines and columns}$$

To simplify the presentation of the stability boundary in two dimensions, the damping factors γ_1, γ_2, and γ_3 are chosen to equal $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. The distance of the tanks (i.e., the difference between the slosh mass locations) from the first tank (rear tank) is considered to be ξ_1 and ξ_2. It is

$$x_1/k = \xi_s \qquad x_2/k = \xi_1 + \xi_s \qquad x_3/k = \xi_2 + \xi_s$$

Representing the stability boundaries in the (ξ_s, γ_s) plane, the Hurwitz determinant $H_7 = 0$ results in

$$\sum_{j=0}^{7} C_j(\xi_s)\gamma_s{}^j = 0$$

where the functions $C_j(\xi_s)$ are polynomials in ξ_s. The stability boundary for the undamped liquid is $C_0(\xi_s) = 0$. It represents the intersection points with the ξ_s axis. For all points (ξ_s, γ_s) above the stability boundary, one obtains stability. Because of $B_n = 0$, the stability is interrupted at the left and at the right. This means that stability is guaranteed only within these boundaries. From $B_8 = 0$, it is recognized that the corresponding stability boundaries to the right and left are given in the form of straight lines perpendicular to the ξ_s axis. These boundaries, however, play no practical role. Substitution of $\xi_s = \gamma_s = 0$ into the Hurwitz determinants determines whether the origin is in the stable or instable region. A necessary and sufficient condition for stability is[4]

1) The coefficients

$$B_n, B_{n-1}, B_{n-3} \dots > 0$$
$$B_1, B_0 > 0 \qquad \text{for even } n$$
$$B_0 > 0 \qquad \text{for odd } n$$

2) The Hurwitz determinant

$$H_{n-1}, H_{n-3} \dots > 0$$
$$H_3 > 0 \qquad \text{if } n \text{ is even}$$
$$H_2 > 0 \qquad \text{if } n \text{ is odd}$$

In the numerical evaluation, the distance of the swivel point from the center of gravity of the vehicle was chosen, x_E = 12.5 m, and the radius of gyration k = 12.5 m. The total length of the space vehicle is 57.5 m. Furthermore, half of the thrust has been considered available for control purposes ($\lambda = \frac{1}{2}$).

A. Rigid Space Vehicle with Propellant Sloshing in One Tank

If the propellant is free to oscillate in only one tank, then the stability polynomial is of fourth degree. This is obtained by removing the last two lines and columns from the determinant. To treat the propellant oscillating in only one tank will present (with small numerical effort) the main influence of the various parameters. It is justified in many vehicles, since the propellant masses in the other tanks are sometimes considerably smaller. It is justified also for the Saturn C-1 vehicle, the booster tanks of which consist of many containers with smaller diameters. Here the sum of these sloshing masses is considerably below the one of the larger second-stage tank. Sometimes, for tanks with lighter propellant (such as liquid hydrogen), the density of which is only a fraction of that of liquid oxygen, the model mass compared with the heavy propellant can be neglected. The stability boundary is obtained from the Hurwitz determinant, $H_{n-1} = 0$ (here $H_3 = 0$ or $B_1B_2B_3 = B_2B_3{}^2 + B_1{}^2B_4$). The points of intersection of this stability boundary with the ξ_s axis are

$$\xi^{(1)} = -|\xi_d| \qquad \xi^{(2)} = |\xi_d|(1-\mu)/a_0\nu_s{}^2$$

where $1/a_0\nu_s{}^2$ is considered to be of small magnitude. This assumption is satisfied in most cases if the control frequency is sufficiently far away from the eigenfrequency of the liquid. The result expresses, therefore, that the stability boundary for small values of $1/a_0\nu_s{}^2$ intersects the ξ_s axis in the vicinity of the center of gravity (origin) and the instantaneous center of rotation $\xi_d = x_d/k$.

One can see that $\xi_2{}^{(2)}$ becomes sensitive toward changes of $1/a_0\nu_s{}^2$. This indicates that, for decreasing gain values, a_0, the intersection point shifts toward the tail of the vehicle. The same behavior occurs if $\nu_s = \omega_s/\omega_c$ decreases. This means that, for decreasing eigenfrequency of the liquid or increasing control frequency, damping has to be applied in the aft section of the vehicle. Figure 4 indicates that the danger zone for instability of the vehicle is located between the center of gravity and the center of instantaneous rotation. In this zone the propellant must be more or less damped, depending on the magnitude of the modal mass of the liquid. For increasing modal mass, more damping is needed in the danger zone to guarantee stability of the vehicle. The vehicle is stable if a container of large modal mass is located outside the danger zone (control frequency being considered small, i.e., ν_s being large). The influence of decreasing frequency ratios ν_s of the propellant eigenfrequency ω_s to the control frequency ω_c can also be found in Fig. 4. If the two frequencies approach each other, the danger zone increases toward the end of the vehicle. Here μ was considered to be one-tenth. The approach of the control frequency toward the liquid frequency or vice versa not only increases the danger zone but also requires more damping to maintain stability. This is most unfavorable if the control frequency is below the eigenfrequency of the propellant, i.e., if $\nu_s < 1$. For $\nu_s > 2$, the wall friction ($\gamma_s = 0.01$) is already sufficient to guarantee stability.

The change of the control damping ζ_c indicates that, for increasing subcritical damping $\zeta_c < 1$, the stability in the danger zone will be diminished whereas, for increased supercritical damping $\zeta_c > 1$, the stability is enhanced. This means that less damping is necessary in the container to maintain stability in the case $\zeta_c > 1$. No additional baffles are required in the danger zone if the mass ratio $\mu = 0.1$ and the control damping $\zeta_c \leq 0.5$ or $\zeta_c \geq 2.0$. This means that, for these values and the parameters $\nu_s = 2.5$ and $a_0 = 3.5$, the wall friction in the container is sufficient to maintain stability.

Another question of great importance in the design of a large space vehicle is the choice of the form of the propellant containers. As seen from Ref. 2, the tank geometry plays an important role for the modal masses and the natural frequencies of the propellant. Containers with large diameters exhibit small natural frequencies that in many cases are too close to the control frequency. The magnitude of the modal mass considerably magnifies this unfavorable effect upon the stability. Clustering of numerous smaller containers not only increases the natural frequencies of the propellant (due to smaller diameters) but also reduces the modal masses, which is a much more important effect. In addition to the weight saving and the slight increase of the natural frequencies, subdivision of tanks by sector walls has the advantage of distributing the modal masses to different vibration modes of the liquid. To summarize, with increasing mass, stability decreases. The influence of the eigenfrequency change of the propellant of fixed modal mass is such that a decrease of the natural frequency increases the danger

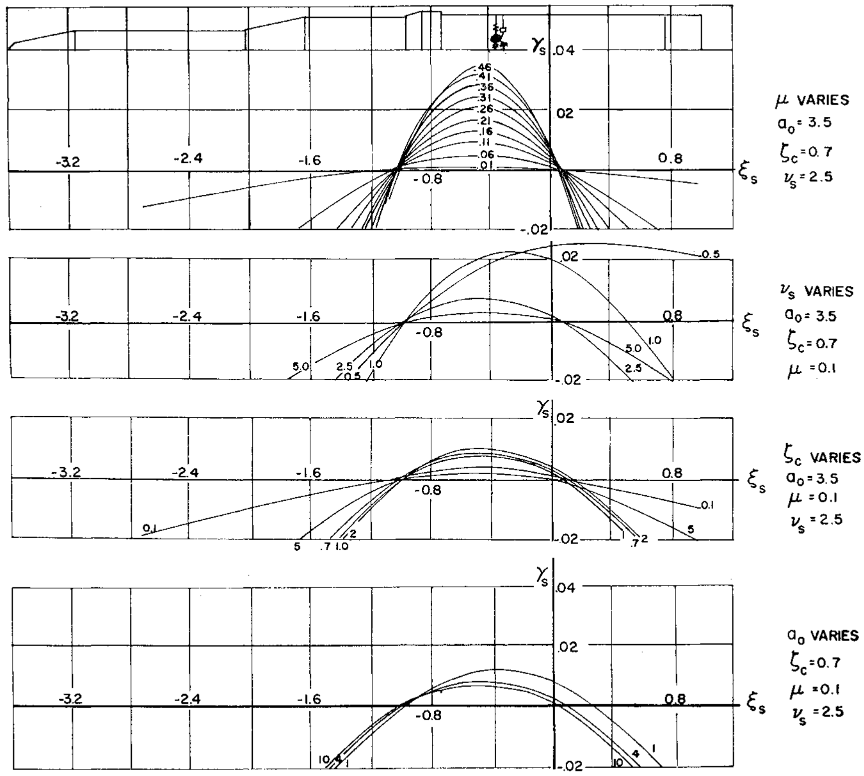

Fig. 4 Stability boundaries for sloshing in one tank

zone toward the end of the vehicle and requires more local damping in the propellant. With increasing natural frequency of the liquid, the influence of the propellant sloshing on the stability of the vehicle diminishes more and more. Wall friction is already sufficient to maintain stability.

The gain value a_0 of the attitude control system shows, for decreasing magnitude, a decrease of stability in addition to a small enlargement of the danger zone toward the end of the vehicle.

In the case of a cluster of tanks with smaller diameters, the results are very similar. The natural frequency is increased due to the smaller diameters. The natural frequency ratio of the propellants in a single barrel circular cylindrical tank of radius a and p, identical circular cylindrical tanks of the same total volume, is

$$\frac{f_n^{(1)}}{f_n^{(p)}} = \frac{1}{(p)^{1/4}} \left\{ \frac{\tanh(\epsilon_n h/a)}{\tanh[\epsilon_n (p)^{1/2} h/a]} \right\}^{1/2}$$

This shows that the frequency increase is proportional only to the slowly increasing value $(p)^{1/4}$. The total sloshing mass, however, decreases rapidly with the inverse value of the square root of the number of containers. The ratio of the total sloshing mass of p tanks and the sloshing mass of the single barrel container is

$$\frac{m_s^{(p)}}{m_s^{(1)}} = \frac{1}{p^{1/2}} \frac{\tanh[\epsilon_n (p)^{1/2} h/a]}{\tanh(\epsilon_n h/a)}$$

This is of great advantage for the dynamics, but from the design and overall performance standpoint, the clustering of tanks has structural and weight disadvantages.

The slosh damping required for p clustered tanks is therefore approximately that of a sloshing mass that is reduced by $1/p^{1/2}$. There is, of course, also a small stability enhancing effect due to the increase of the natural frequency.

B. Rigid Space Vehicle with Propellant Sloshing in Two Tanks

In some cases the influence of the propellant in a second tank cannot be neglected, making the determination of stability boundaries for vehicles with two sloshing masses mandatory. It is obtained by removing the last column and line from the determinant (5).

1. *Two concentric containers*

It may be possible to remedy the influence of propellant sloshing by choosing a concentric tank arrangement consisting of an inner tank with circular cross section (radius $r = b$) and an outer tank of annular cross section with an outer radius $r = a$. By proper choice of the diameter ratio $k = b/a$, the liquid masses in the inside container and outside container can be brought into such a phase relation that the forces and moments of these individual tanks cancel each other. Figures 5, 6, and 7 show the results of this study for diameter ratios $k = b/a = 0.3$, 0.5, and 0.7. The results are very similar to those of case IIIA since the difference in the location of the sloshing masses is very small. The danger zone is increased somewhat toward the rear of the vehicle. For increasing control damping ζ_c, the stability is decreased. Increase of the control frequency enhances the stability as in IIIA. For a sloshing frequency of the center tank below the control frequency, more baffling has to be employed over the enlarged danger zone. For increasing sloshing frequency the stability increases. To obtain maximum cancellation effects, the sloshing masses of the center and outer container should be equal and should oscillate in antiphase. Equal sloshing masses can be obtained for a diameter ratio of about $k = 0.77$ for which, unfortunately, the phases are not favorable. If the phases are chosen favorably, as in the case of a diameter ratio $k = 0.5$, the sloshing masses exhibit a ratio of 1:5. This shows that no pronounced benefit can be obtained by a concentric tank arrangement. For the diameter ratios $k = 0.3$, 0.5, and 0.7, the damping required for stability in the container is in the ratio 12:9:8.

2. *Quarter tank arrangement*

As Fig. 3 shows, compartmentation of containers by radial walls exhibits considerably decreased sloshing masses. In the case of a four-quarter tank arrangement, the first modal mass is only about a third of the value of a cylindrical container with circular cross section. But other vibration modes are still important, such as the consecutive sloshing

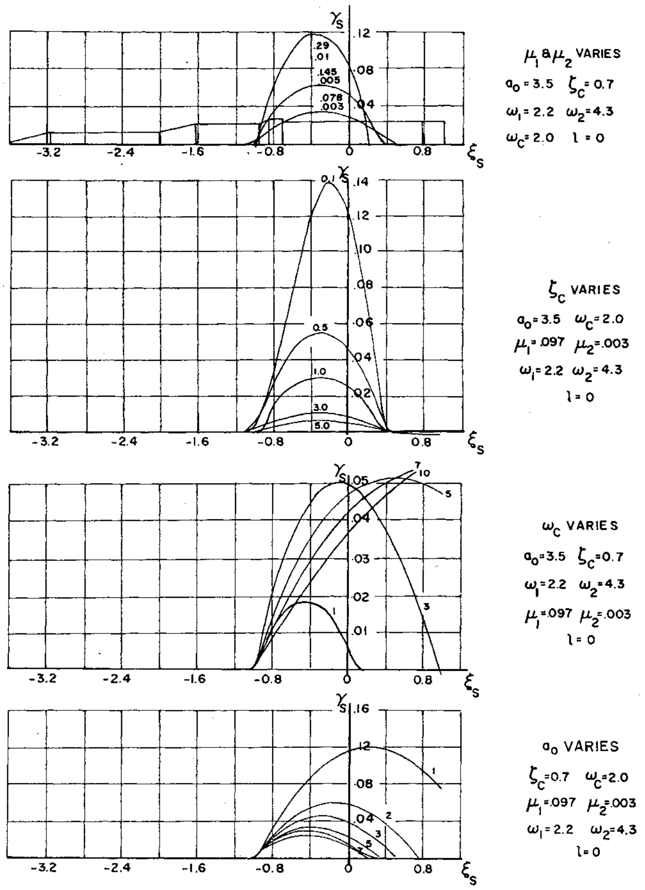

Fig. 5 Stability boundaries for sloshing in concentric tanks (k = 0.3)

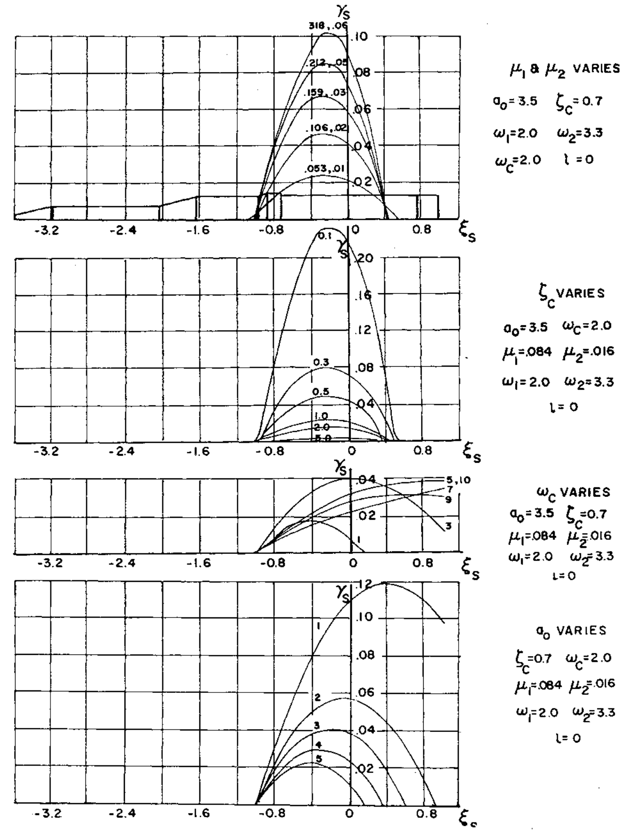

Fig. 6 Stability boundaries for concentric tanks (k = 0.5)

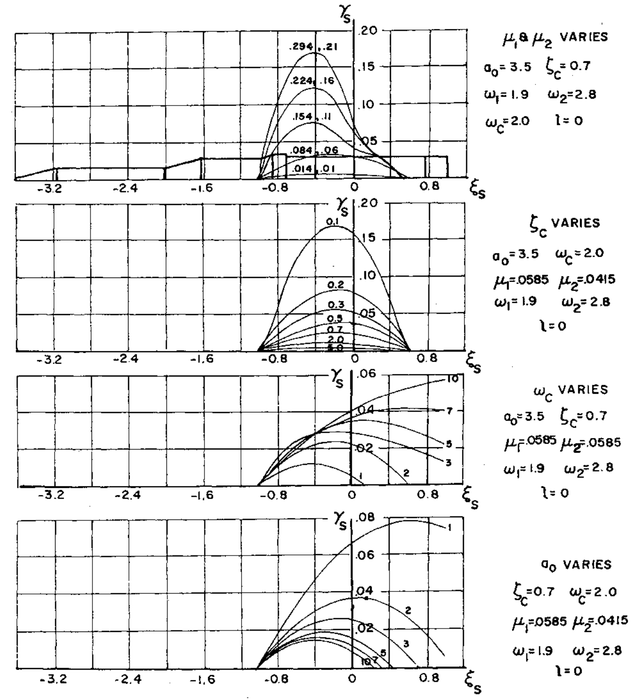

Fig. 7 Stability boundaries for concentric tanks (k = 0.7)

mode of which the mass still represents 43% of that of the first one. This indicates that in stability investigations this second mode can no longer be neglected. This tank arrangement has, besides the reduced modal mass, the advantage that the fundamental frequency is slightly larger than that of a container of circular cross section (Fig. 2). Also, the total sloshing mass is distributed to various modes; thus it is not all excited at the same frequency as in clustered tanks.

In the equations of motion, two slosh masses are again considered, representing the first and second sloshing mode of the quarter tank arrangement. The results are very similar to the previous ones. Again the danger zone is located between the center of instantaneous rotation and the center of gravity. The increase of the control damping ζ_c in the

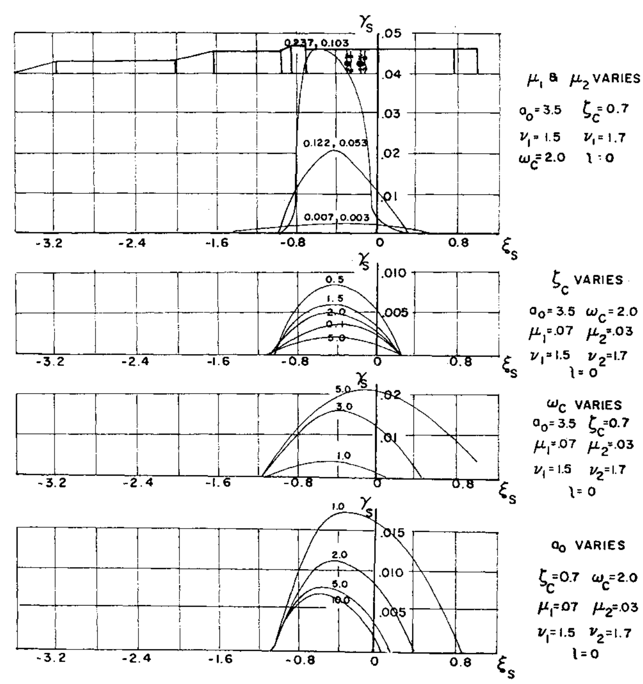

Fig. 8 Stability boundaries for sloshing in quarter tank arrangement

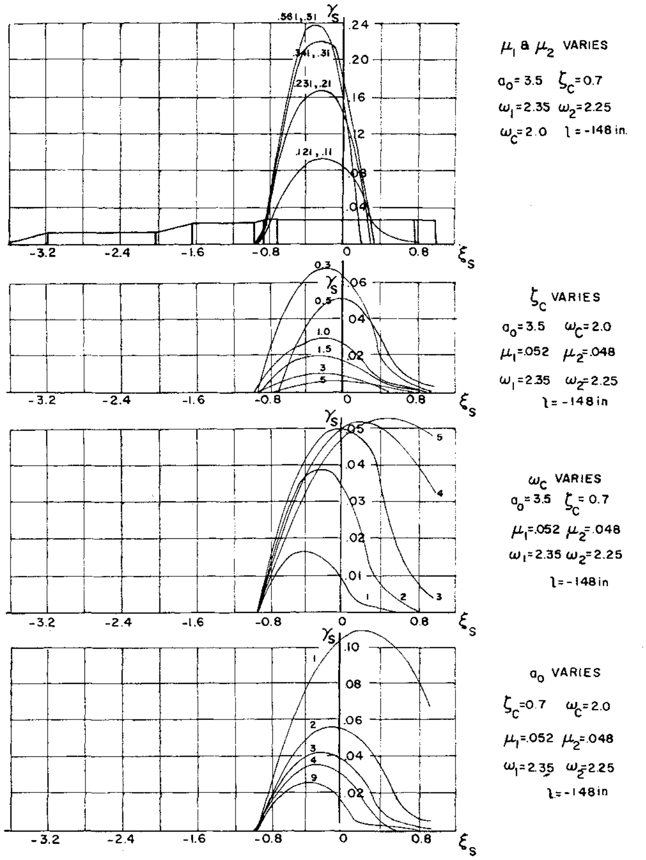

Fig. 9 Stability boundaries for sloshing in two tanks in tandem arrangement

subcritical region decreases the stability, while increase in the supercritical region increases the stability region. For increasing control frequency, the danger zone is enlarged toward the base of the vehicle and requires larger damping in the tank to maintain stability. The influence of the simultaneous change of the sloshing frequencies is exhibited. Sloshing frequencies below the control frequency again require more baffling in an enlarged danger zone (toward the rear). Increase of sloshing frequencies results in a decrease of the danger zone toward the one between the center of gravity and center of instantaneous rotation and requires less damping in the tanks to maintain stability of the vehicle. Low gain values ($a_0 = 1$) require more baffling along a larger danger zone, while increase of the gain a_0 reduces the danger zone and the requirement of damping in the tanks (Fig. 8).

3. *Tandem arrangement of two tanks*

For a tandem tank arrangement, the results can be seen in Fig. 9. Tank number 1 is designated as the rear tank, and number 2 is the forward tank. The distance between the two sloshing masses is called ξ_1. The sloshing frequency of the liquid in these two tanks is the same. In the numerical evaluation, the diameter was taken to be 256 in. It can be seen that for increasing sloshing mass, a loss in stability region is encountered. Furthermore, the danger zone is shifted slightly toward the rear of the craft. Increasing control damping ζ_c increases the stability area. An increase of the control frequency ω_c results in an increased baffling requirement over an enlarged danger zone toward the aft of the craft.

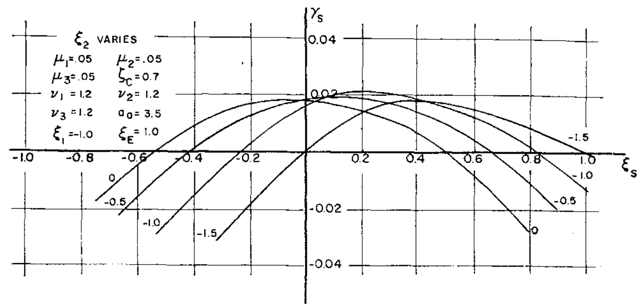

Fig. 10 Stability boundaries for sloshing in three tanks in tandem arrangement

For sloshing frequencies below the control frequency, a large amount of damping is required in the enlarged danger zone to maintain stability. Increase of the sloshing frequency decreases the danger zone and the required amount of damping.

The change of the gain value a_0 has an effect similar to that for a single tank. Small gain values require strong baffling over an enlarged danger zone. Increase in the gain value enlarges the stability region and shifts the danger zone slightly toward the center of instantaneous rotation.

The influence of the difference, ξ_1, in the tank locations exhibits, for increasing distance between tanks, a shifting of the danger zone toward the aft of the vehicle with slightly less damping requirements. This indicates that, in a vehicle in which one sloshing mass is stationary during flight time and the other liquid mass shifts toward the aft of the vehicle, the entire rear part of the vehicle has to be provided with appropriate damping to maintain stability (Fig. 9).

C. Rigid Space Vehicle with Propellant Sloshing in Three Tanks

In almost all space vehicles, the consideration of sloshing in three propellant containers is sufficient for simplified stability boundary determinations. The liquid propellants of the remaining tanks exhibit either small sloshing masses (due to their low density or different tank geometry) or larger natural frequencies of the propellants. For this case the total determinant (5) is treated. The results are similar as in the tandem arrangement of two tanks. Figure 10 exhibits the shifting (ξ_2) of the two booster slosh masses toward the rear of the space vehicle (as it takes place during the draining of the first stage containers). The danger zone shifts with increasing slosh mass difference aft of the vehicle. This again indicates that the booster must be provided with appropriate baffles to maintain stability.

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